LA-7811-MS

Informal Report

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This report was not edited by the Technical Information Staff.

This work was supported by the US Department of Energy, Division of Basic Energy Sciences.

UNITED STATES
DEPARTMENT OF ENERGY
CONTRACT W-7405-ENG. 36

LA-7811-MS
Informal Report
UC-25
Issued: May 1979

An Analytic Method for Calculating the Time-Temperature History of Metal Foils Under Pulsed Irradiation and a Gaussian Beam Profile

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AN ANALYTIC METHOD FOR CALCULATING THE TIME-TEMPERATURE HISTORY OF METAL FOILS UNDER PULSED IRRADIATION AND A GAUSSIAN BEAM PROFILE

by

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ABSTRACT

Utilization of a pulsed radiation source such as the Clinton P. Anderson Meson Physics Facility (LAMPF) for materials science studies requires knowledge of the time-temperature history of a subject metal foil. We derive an analytic solution to a two-dimensional heat flow equation, incorporating the LAMPF time structure and the LAMPF Gaussian beam spot profile. This calculational method is useful in designing experimental systems for materials science studies and can be done on a Hewlett-Packard model 97 desk-top calculator. We compare the results with an equivalent numerical solution of the same two-dimensional heat flow problem done on a digital computer.

I. INTRODUCTION

We utilize the 800-MeV proton beam at LAMPF as a source for radiation damage, materials science studies. Materials phenomenon are strongly temperature dependent. The calculation described here is used in the design of experimental systems to give a theoretical prediction of the temperature history of a subject foil. Since large temperature excursions during a pulse are not usable conditions for an experiment, we consider constant physical properties, near the design point in temperature, for the materials under study. Although our calculations reflect the LAMPF time structure (0.5 ms "on time" at 120 Hz), any time structure may be analyzed by insertion of the proper values of τ, the time between pulses, and τ_1, the pulse length. We also incorporate a Gaussian beam profile; typical of the LAMPF beam. We approximate elliptical beam spots by an equivalent circular area. We assume that a coolant such as flowing water will maintain a constant temperature (coolant temperature plus film temperature gradient) at the surface of an emersed foil and at a radius of 3σ; σ is one standard deviation. The calculation can be done for any material for which the physical properties and particle energy dissipation characteristics are known.

II. EQUATIONS

The two-dimensional time-dependent heat conduction equation in cylindrical coordinates is

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \frac{\partial^2 T}{\partial z^2} = \frac{1}{K} Q \; . \qquad (1)$$

The region of interest is a finite cylinder of height ℓ and radius a, whose surface is held at a fixed temperature $T_o = T(t=0)$ (Figure 1). Treating the right-hand side of equation (1) as a source term allows us to write the formal solution in terms of the Green's function for the homogeneous equation as[1]

$$T(r,z,t) = \frac{\kappa}{K}\int_o^a \int_o^\ell \int_o^t G(r,r';z,z';t,t')Q(r',z',t')\; 2\pi r' dr' dz' dt' . \qquad (2)$$

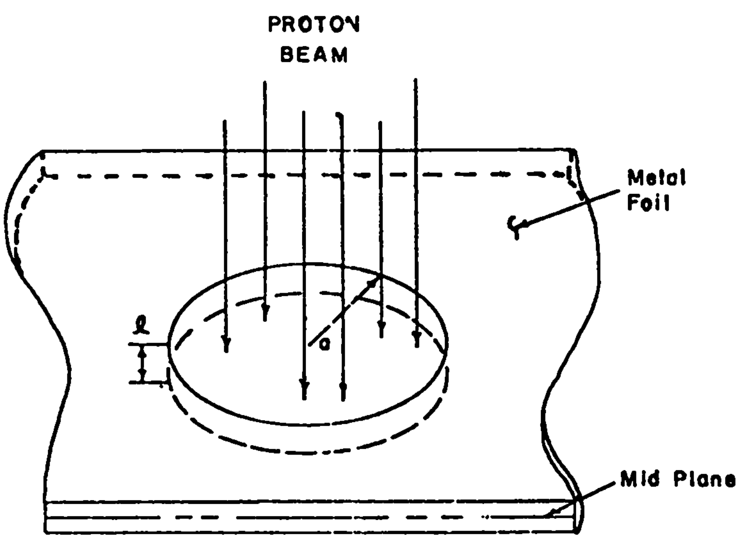

Fig. 1.
Foil geometry.

To find the necessary Green's function we use the method of separation of variables on the modified heat conduction equation

$$\frac{1}{\kappa}\frac{\partial G}{\partial t'} - \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial G}{\partial r'}\right) - \frac{\partial^2 G}{\partial z'^2} = 0 . \tag{3}$$

Assuming $G(r',z',t') = R(r')Z(z')\Theta(t')$ gives

$$0 = \frac{1}{\kappa\Theta(t')}\frac{\partial\Theta(t')}{\partial t'} - \frac{1}{r'R(r')}\frac{\partial}{\partial r'}\left(r'\frac{dR(r')}{dr'}\right) - \frac{1}{Z(z')}\frac{d^2Z(z')}{dz'^2} , \tag{4}$$

which breaks up into the three ordinary differential equations

$$\frac{d\Theta(t')}{dt'} = -(c_r^2 + c_z^2)\kappa\Theta(t') \tag{5}$$

$$\frac{d^2Z(z')}{dz'^2} = -c_z^2\, Z(z') \tag{6}$$

$$\frac{1}{r'}\frac{d}{dr'}\left(r'\frac{dR(r')}{dr'}\right) = -c_r^2\, R(r') . \tag{7}$$

Equation (7) is the standard equation obeyed by Bessel functions of order zero

$$r^{*2}\frac{d^2R}{dr^{*2}} + r^*\frac{dR}{dr^*} + (r^{*2} - (n=0)^2)\, R = 0 , \tag{8}$$

where $r^* = c_r r'$. The boundary condition on this radial equation is

$$T\,(a,z,t) = 0 \text{ (or a constant)} , \tag{9}$$

giving immediately $r' = a \leftrightarrow r^* = \alpha_n$, where α_n is the nth zero of $J_o(r^*)$, so that

$$c_r = \frac{\alpha_n}{a} . \tag{10}$$

(The zeroeth order Bessel function of the second kind is eliminated since it cannot satisfy boundedness at $r = 0$.) The solution to the radial equation is then

$$R(r') = \sum_{\alpha_n} A_n J_o \left(\alpha_n \frac{r'}{a}\right) , \tag{11}$$

where the coefficient A_n is determined from the conditions

$$R(r' = r) = \delta(r'-r) = f(r') \tag{12}$$

to be

$$f(r') = \sum_{\alpha_n} A_n J_o \left(\alpha_n \frac{r'}{a}\right) ,$$

$$\int_o^a 2\pi r' dr' J_o \left(\alpha_m \frac{r'}{a}\right) \delta(r'-r) = \int_o^a 2\pi r' dr' \sum_{\alpha_n} A_n J_o \left(\alpha_n \frac{r'}{a}\right) J_o \left(\alpha_m \frac{r'}{a}\right) ,$$

$$J_o \left(\alpha_m \frac{r}{a}\right) = \pi a^2 A_m \left[J_1^* (\alpha_m)\right]^2 ,$$

$$A_m = \frac{1}{\pi a^2} \frac{J_o(\alpha_m \frac{r}{a})}{\left[J_o'(\alpha_m)\right]^2} , \tag{13}$$

giving finally

$$R(r,r') = \frac{1}{\pi a^2} \sum_{\alpha_n} \frac{J_o(\alpha_n \frac{r'}{a}) \, J_o(\alpha_n \frac{r}{a})}{\left[J_o' (\alpha_n)\right]^2} . \tag{14}$$

Equation (6) is even simpler in that the solutions are trigonometric rather than Bessel Functions. Using the boundary condition

$$Z(z' = 0) = Z(z' = \ell) = 0 \tag{15}$$

gives the solution

$$Z(z') = \sum_m B_m \sin\left(\frac{m\pi z}{\ell}\right), \tag{16}$$

where the cosines disappear because of the $z = 0$ condition, and the $z = \ell$ condition gives

$$c_z \ell = m\pi \text{ or } c_z = \frac{m\pi}{\ell}. \tag{17}$$

The coefficients B_m are evaluated using

$$Z(z'=z) = \delta(z-z') = f(z') \tag{18}$$

to give

$$\int_o^\ell \sin\left(\frac{m\pi z}{\ell}\right) \delta(z-z')dz' = \int_o^\ell \sum_m A_m \sin\left(\frac{m\pi z'}{\ell}\right) \sin\left(\frac{m\pi z'}{\ell}\right) dz'$$

$$\sin\left(\frac{n\pi z}{\ell}\right) = \frac{\ell}{2} A_n \tag{19}$$

$$A_n = \frac{2}{\ell} \sin\left(\frac{n}{\ell} z\right)$$

giving finally

$$Z(z,z') = \frac{2}{\ell} \sum_{m} \sin\left(\frac{m\pi z'}{\ell}\right) \sin\left(\frac{m\pi z}{\ell}\right). \tag{20}$$

Utilizing the results of equations (10) and (17) allows equation (5) to be written as

$$\frac{d\theta(t')}{dt'} = -\kappa\left(\frac{\alpha_n^2}{a^2} + \frac{(m\pi)^2}{\ell^2}\right) \theta(t'), \tag{21}$$

whose solution is

$$(t,t') = \sum_{n}\sum_{m} \exp\left\{-\kappa\left[\frac{\alpha_n^2}{a^2} + \frac{(m\pi)^2}{\ell^2}\right] (t-t')\right\}. \tag{22}$$

Thus the Green's function for this problem is

$$G(r,r';z,z';t,t') = \frac{2}{\pi a^2 \ell} \sum_{m=1}^{\infty} \exp\left[-\kappa \frac{m^2\pi^2}{\ell^2} (t-t')\right] \sin\left(\frac{m\pi z}{\ell}\right) \sin\left(\frac{m\pi z'}{\ell}\right)$$

$$\times \sum_{n} \exp\left[-\kappa \frac{\alpha_n^2}{a^2} (t-t')\right] \frac{J_o(\alpha_n \frac{r}{a})\, J_o(\alpha_n \frac{r'}{a})}{\left[J_o'(\alpha_n)\right]^2}. \tag{23}$$

III. APPLICATION

The LAMPF 800-MeV pulsed proton beam has an elliptically shaped target area approximated here by a circular area of radius a. Irradiation and heating of a thin rectangular foil whose edges and surface are held constant at the initial temperature can be approximated by considering heat conduction and generation in a finite cylinder large enough that the radial edges see little radiation. The proton flux for the LAMPF beam is Gaussian in profile and hence can be described as

$$I(r,z,t) = I(t) \exp\left(-\frac{r^2}{r_o^2}\right), \tag{24}$$

where r_o is a constant ($= \sqrt{2}$ where σ is the standard deviation of the Gaussian profile) giving the average size of the beam spot and I(t) is a pulsed time function equal to

$$I(t) = \begin{cases} 0 & \text{for } t < 0 \\ I_o & \text{for } m\tau \le t \le m\tau + \tau_1 \\ 0 & \text{for } m\tau + \tau_1 \le t \le (m+1)\tau \end{cases} , \tag{25}$$

where $m = 0, 1, 2, \ldots, \tau_1 = 0.5$ ms and $\tau = \frac{1}{120\ \text{Hz}}$. For a net current of 1 mA, I_o is given by

$$I_o = \frac{n_p/\Delta t}{\int_o^a 2\pi r \exp\left(-\frac{r^2}{r_o^2}\right) dr} \cong \frac{n_p/\Delta t}{\pi r_o^2} , \tag{26}$$

where n_p is the total number of protons in a pulse of duration Δt, given by

$$\begin{aligned} n_p &= 1\ \text{mA} \times \frac{10^{-3}\ \text{C}}{\text{sec-mA}} \times 6.28\ 10^{18}\ \frac{\text{p}}{\text{C}} \times \frac{1}{120\ \text{pulse/sec}} \\ &= 5.233\ 10^{13}\ \frac{\text{p}}{\text{pulse}} \end{aligned} \tag{27}$$

for a net current of 1 mA. If each 800-MeV proton loses an amount of energy ε^* per unit distance traversed in the target, and if all this energy is assumed to produce heat by direct excitation of the lattice, the heat generation rate per unit volume is

$$Q = \varepsilon^* \frac{n_p/\Delta t}{\pi r_o^2} \exp\left[-\frac{r^2}{r_o^2}\right] \tag{28}$$

whenever the beam is on, and zero otherwise. To insure that most of the beam is accounted for we set $a = 3\sigma = 2.12132 r_o$ (by which time we are using 99% of the current).

The temperature is given by equations (2) and (23) as

$$T(r,z,t)-T_o = \frac{\kappa}{K} Q \frac{2}{\pi a^2 \ell} \sum_{m=1}^{\infty} \sum_{\alpha_n} \int_o^t \exp\left[-\kappa\left(\frac{m^2\pi^2}{\ell^2}+\frac{\alpha_n^2}{a^2}\right)(t-t')\right] f(t')dt' \tag{29}$$

$$\times \int_o^a 2\pi r' \exp\left(-\frac{r'^2}{r_o^2}\right)\frac{J_o(\alpha_n \frac{r}{a})J_o(\alpha_n \frac{r'}{a})}{[J_o'(\alpha_n)]^2} dr' \int_o^\ell \sin\left(\frac{m\pi z}{\ell}\right)\sin\left(\frac{m\pi z'}{\ell}\right)dz',$$

where $f(t')$ is a pulsed time function such as given by equation (25) but normalized to unity.

The integration in z' is readily carried out to give

$$\int_o^\ell \sin\left(\frac{m\pi z}{\ell}\right)\sin\left(\frac{m\pi z'}{\ell}\right)dz' = \frac{2\ell}{m\pi}\sin\left(\frac{m\pi z}{\ell}\right) \quad m = 1,3,5,\ \ldots \tag{30}$$

and the summation over odd m can be rewritten as $m \to 2m+1$. The integration in r' can only be approximated, using the definite integral

$$\int_o^\infty r^{\nu+1} e^{-a^2r^2} J_\nu(br)dr = \frac{b^\nu}{(2a)^{\nu+1}}\exp\left[-\frac{b^2}{4a^2}\right] \tag{31}$$

to give

$$\int_o^a 2\pi r' \frac{J_o(\alpha_n \frac{r'}{a})J_o(\alpha_n \frac{r}{a})}{\left[J_o'(\alpha_n)\right]^2}\exp\left[-\frac{r'^2}{r_o^2}\right]dr' \simeq 2\pi\frac{J_o(\alpha_n \frac{r}{a})}{J_1^2(\alpha_n)}\sigma^2\exp\left[-\frac{\alpha_n^2}{18}\right]$$

$$\simeq \pi r_o^2 \frac{J_o(\alpha_n \frac{r}{a})}{J_1^2(\alpha_n)}\exp\left(-\frac{\alpha_n^2}{18}\right), \tag{32}$$

where we have used the relations $J_o'(\alpha_n) = J_1(\alpha_n)$ and $2\sigma^2 = r_o^2 = \frac{a^2}{4.5}$. Due to the rapidly decaying nature of the Gaussian the approximation $a \approx \infty$ in the integration limit is quite reasonable.

The integration in t' is done by defining the integral

$$I_{\alpha m} = \beta_{\alpha m} \int_o^t \exp(\beta_{\alpha m} t')\, f(t')dt', \tag{33}$$

where

$$\beta_{\alpha m} = \left(\frac{\alpha_n^2}{a^2} + \frac{(2m+1)^2\pi^2}{\ell^2} \right). \tag{34}$$

Since $f(t')$ only exists for $k\tau \le t \le k\tau + \tau_1$ the integral $I_{\alpha m}$ reduces to

$$\begin{aligned} I_{\alpha m} = {} & \beta_{\alpha m} \int_o^{\tau_1} \exp(\beta_{\alpha m} t')dt' + \int_\tau^{\tau+\tau_1} \beta_{\alpha m} \exp(\beta_{\alpha m} t')dt + \ldots + \\ & + \beta_{\alpha m} \int_{(k-1)\tau}^{(k-1)\tau+\tau_1} \exp(\beta_{\alpha m} t')dt' \\ & + \int_{k\tau}^{t \text{ or } k\tau+\tau_1} \beta_{\alpha m} \exp(\beta_{\alpha m} t')dt'. \end{aligned} \tag{35}$$

The lower limits give the sum

$$- 1 - \exp(\beta_{\alpha m}\tau) - \exp(2\beta_{\alpha m}\tau) - \ldots - \exp(k\beta_{\alpha m}\tau) = \frac{\exp[(k+1)\beta_{\alpha m}\tau]-1}{1 - \exp(\beta_{\alpha m}\tau)}, \tag{36}$$

while the upper limits give

$$\exp(\beta_{\alpha m}\tau_1) \left| 1+\exp(\beta_{\alpha m}\tau) + \exp(\beta_{\alpha m}2\tau) + \ldots + \exp\left[(k-1)\beta_{\alpha m}\tau)\right] \right|$$

$$+ \begin{cases} \exp\,(\beta_{\alpha m}\tau) \\ \exp\,[\beta_{\alpha m}\,(k\tau+\tau_1)] \end{cases} \tag{37}$$

$$= \exp\,(\beta_{\alpha m}\tau_1)\left[\frac{\exp\,(k\beta_{\alpha m}\tau)-1}{\exp\,(\beta_{\alpha m}\tau)-1}\right] + \begin{cases} \exp\,(\beta_{\alpha m}t) \\ \exp\,[\beta_{\alpha m}(k\tau+\tau_1)]\ . \end{cases} \tag{38}$$

Combining these results yields

$$I_{\alpha m} = \frac{1-\exp\,[(k+1)\beta_{\alpha m}\tau] + \exp\,[\beta_{\alpha m}(k\tau+\tau_1)] - \exp\,(\beta_{\alpha m}\tau_1)}{\exp\,(\beta_{\alpha m}\tau) - 1}$$

$$+ \begin{cases} \exp\,(\beta_{\alpha m}\tau) \\ \exp\,[\beta_{\alpha m}\,(k\tau+\tau_1)]\ , \end{cases}$$

where the two final terms refer respectively to times inside and between pulses.

Putting together equations (29), (30), (32) and (38) gives as the entire solution

$$T(r,z,t) - T_o = \frac{\kappa}{K}\,Q\,\frac{2}{\pi a^2 \ell} \sum_m \left[\frac{2\ell}{(2m+1)\pi}\ \sin\left(\frac{(2m+1)\pi z}{\ell}\right)\right] \sum_\alpha \left[\pi r_o^2 \exp\left(-\frac{\alpha_n^2}{18}\right)\frac{J_o(\alpha_n \frac{r}{a})}{J_1^2\ (\alpha_n)}\right]$$

$$\times \left[\frac{1}{\kappa}\left(\frac{a^2\ell^2}{\alpha_n^2\ell^2+(2m+1)^2\pi^2a^2}\right)\exp\,(-\beta_{\alpha m}t)\left\{\frac{1-\exp[(k+1)\beta_{\alpha m}\tau]+\exp[\beta_{\alpha m}(k\tau+\tau_1)]-\exp(\beta_{\alpha m}\tau_1)}{\exp\,(\beta_{\alpha m}\tau)-1} + \exp(\beta_{\alpha m}t)\right\}\right] \tag{39a}$$

for $k\tau \le t \le k\tau+\tau_1$

$$T(r,z,t)-T_o = \frac{\kappa}{K} Q \frac{2}{\pi a^2 \ell} \sum_m \left[\frac{2\ell}{(2m+1)\pi} \sin\left(\frac{(2m+1)\pi z}{\ell}\right)\right] \sum_\alpha \left[\pi r_o^2 \exp\left(-\frac{\alpha_n^2}{18}\right) \frac{J_o(\alpha_n \frac{r}{a})}{J_1^2(\alpha_n)}\right] \times$$

$$\left[\frac{1}{\kappa}\left(\frac{a^2\ell^2}{\alpha_n^2\ell^2+(2m+1)^2\pi^2a^2}\right) \exp(-\beta_{\alpha m} t) \left\{\frac{1-\exp[(k+1)\beta_{\alpha m}\tau]+\exp[\beta_{\alpha m}(k\tau+\tau_1)]-\exp(\beta_{\alpha m}\tau_1)}{\exp(\beta_{\alpha m}\tau)-1} + \exp\left[\beta_{\alpha m}(k\tau+\tau_1)\right]\right\}\right] \quad (39b)$$

for $k\tau+\tau_1 \le t \le (k+1)\tau$.

Rearranging the constants and returning to the $I_{\alpha m}$ notation gives

$$T(r,z,t) = T_o + \frac{Q}{K}\frac{8}{9\pi} \sum_m \frac{\sin \frac{(2m+1)\pi z}{\ell}}{(2m+1)} \sum_{\alpha_n} \exp\left(-\frac{\alpha_n^2}{18}\right)\frac{J_o(\alpha_n \frac{r}{a})}{J_1^2(\alpha_n)} \frac{\kappa}{\beta_{am}} \exp(-\beta_{\alpha m} t) I_{\alpha m} \cdot \quad (39c)$$

IV. RESULTS

We have used this calculational method for evaluating various irradiation locations and experimental systems at LAMPF. Figure 2 is a schematic of a typical result. This plot represents the temperature-time history at the mid-plane of a metal foil, at the center of the Gaussian beam spot. This result generalizes and may be extended to an entire family of (r,z) positions which gives the total temperature profile of the foil at a given time.

The calculation had been done previously by numerical methods on a digital computer. A comparison of the results from the two methods is shown in Table I. Run time on a HP-97 calculator for a given time and position (r,z) is approximately 4 min.

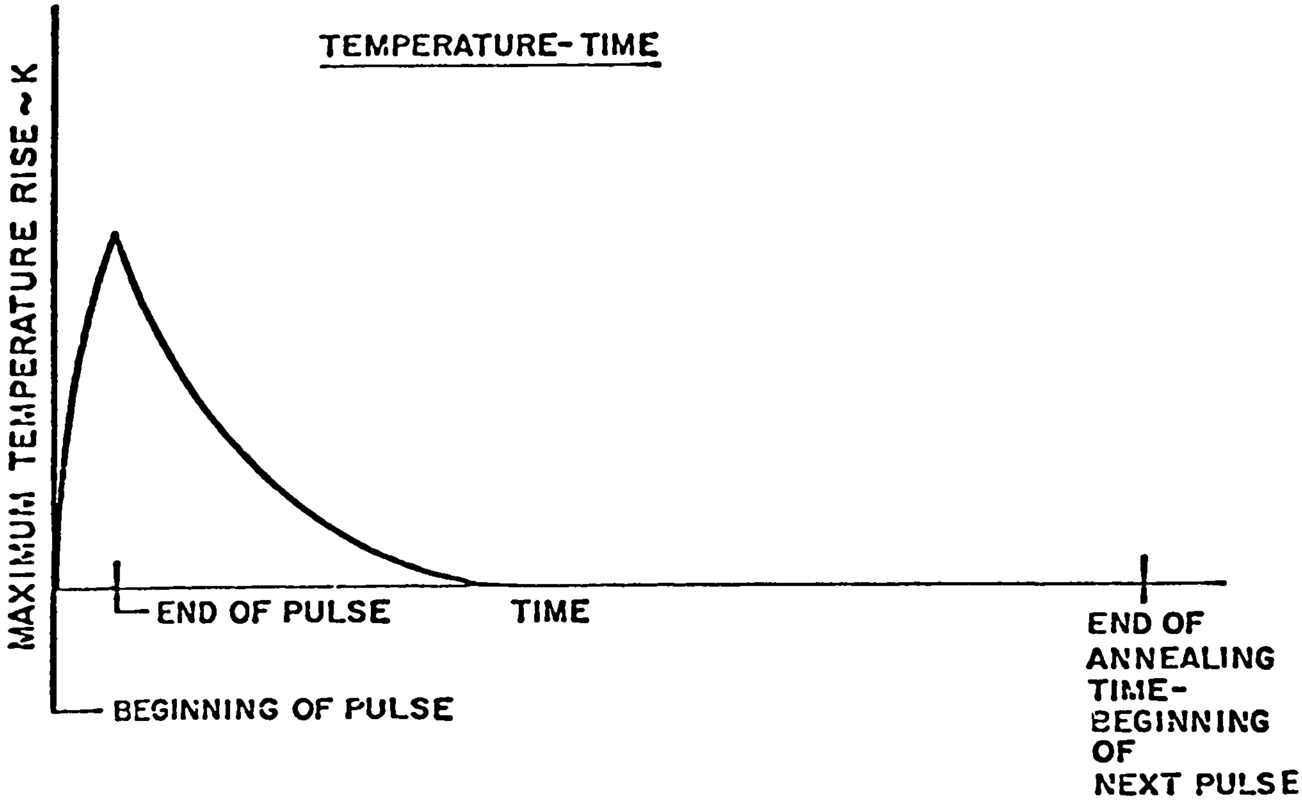

Figure 2. Schematic of the temperature - time profile of a metal foil under pulsed irradiation at the foil mid-plane and at the center of a Gaussian beam spot.

TABLE I

COMPARISON OF THE ANALYTIC AND NUMERICAL TEMPERATURE CALCULATION

Material		Aluminum
ℓ	-	1×10^{-3}m
r_o	-	1.94×10^{-3}m
I_o	-	8.87×10^{21}pm^{-2}s^{-1}
r	-	0
T_o	-	400 K

Time	Temperature	
	Numerical	Calculator
5×10^{-4}	535.304	535.338
1×10^{-3}	494.003	491.702
2.8×10^{-3}	418.764	416.742
8.3×10^{-3}	400.154	400.095

ACKNOWLEGEMENT

We thank W. V. Green for his encouragement in this endeavor.

REFERENCE

1. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, Second Edition (Clarendon Press Oxford, 1959), pp. 21 and 368.

Printed in the United States of America. Available from
National Technical Information Service
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